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General theory of the matrix formulation of the automorphisms of affine Kac–Moody algebras

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Abstract. The determination of conjugacy classes within the group of all automorphisms of an affine Kac–Moody algebra is discussed in detail, and the limited role of Cartan preserving automorphisms is emphasized. A comprehensive method of dealing with all the automorphisms of an untwisted affine Kac–Moody algebra is developed. This is based on a matrix formulation of the untwisted affine Kac–Moody algebras. Four different types of automorphism (called 'type 1a', 'type 1b', 'type 2a' and 'type 2b') are identified and analysed within this matrix formulation. The development of the detailed properties of automorphisms in the matrix formulation includes formulae for the products and inverses of automorphisms, together with the conjugacy and involutive conditions.

1. Introduction

This is the first of a series of papers which will deal with the problem of determining the conjugacy classes of the group of automorphisms of an affine Kac–Moody algebra. The important role played by the automorphism groups of Lie algebras is well known. In particular, the study of the involutive automorphisms of complex semisimple Lie algebras by Gantmacher [1] allowed Gantmacher [2] to obtain a very elegant systematic determination of all the simple real Lie algebras.

This first paper is devoted to an examination of the general questions involved. This starts, in section 2, with a study of the subgroup of 'Cartan preserving automorphisms' of an affine Kac–Moody algebra. However, although such automorphisms are very important, in that every conjugacy class of the automorphism group contains at least one Cartan preserving automorphism, it is necessary to go beyond such automorphisms. The main reason for this is that it is possible for two Cartan preserving automorphisms to be conjugate members of the group of all automorphisms of an affine Kac–Moody algebra, even though they are not conjugate within the subgroup of Cartan preserving automorphisms. That is, conjugacy of Cartan preserving automorphisms within the group of all automorphisms of an affine Kac–Moody algebra is often achieved via 'non-Cartan preserving automorphisms'.

To enable this problem to be tackled systematically a comprehensive method of dealing with all the automorphisms of an untwisted affine Kac–Moody algebra is presented in section 3. This is based on a matrix formulation of the untwisted affine Kac–Moody algebras. As will be shown in section 3, in general there are four types of automorphism within this matrix formulation. These will be called 'type 1a', 'type

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1b', type 2a' and 'type 2b'. Section 3 contains the explicit derivation of all of these types of automorphism, as well as the essential motivational ideas.

The detailed properties of automorphisms within this matrix formulation are developed in section 4. This includes:

(i) an investigation of identical automorphisms and of the identity automorphism,
(ii) formulae for the products of automorphisms;
(iii) the conditions for an automorphism to be involutive;
(iv) formulae for the inverses of automorphisms;
(v) the conjugacy conditions for automorphisms;
(vi) some general remarks on the conjugacy classes of involutive automorphisms.

The second paper of this of this series will contain a detailed application of this general theory to the special case of the affine Kac–Moody algebra $A_1^{(1)}$, with particular attention being paid to the involutive automorphisms. As $A_1^{(1)}$ is the 'simplest' untwisted affine Kac–Moody algebra it provides a perfect example for testing the practical applicability of the concepts of the present paper, and for comparing the matrix method with more traditional structural techniques. In subsequent papers this analysis will be extended first to $A_2^{(1)}$ and then to $A_l^{(1)}$ for values of l greater than 2. (It should be noted that $A_1^{(1)}$ has special features that are absent in $A_l^{(1)}$ with $l>1$, so $A_1^{(1)}$ has to be studied separately from the rest of the $A_l^{(1)}$ family. Moreover, as the algebras $A_l^{(1)}$ with $l>2$ are quite complicated, it is also worth while treating the case $A_2^{(1)}$ separately.)

It is intended to examine the groups of automorphisms of other affine Kac–Moody algebras (including those of the 'twisted' variety) at a later date.

The structure of the affine Kac–Moody algebras and their Weyl groups is now well known (cf Kac [3] and Cornwell [4]). Unless otherwise stated, all the notations and conventions that will be employed in the present paper are those of Cornwell [4]. In particular, quantities belonging to the simple complex Lie algebra $\tilde{\mathscr{L}}^0$ associated with an untwisted affine Kac–Moody algebra $\tilde{\mathscr{L}}$ are distinguished from the corresponding quantities belonging to $\tilde{\mathscr{L}}$ by a superscript 0, so that, for example, α is the linear functional on the Cartan subalgebra $\mathscr{H}$ of $\tilde{\mathscr{L}}$ that is the extension of the linear functional α^0 on the Cartan subalgebra $\mathscr{H}^0$ of $\tilde{L}^0$.

2. Cartan preserving automorphisms

An automorphism of an affine Kac–Moody algebra $\tilde{\mathscr{L}}$ that maps every element of its Cartan subalgebra $\mathscr{H}$ into an element of $\mathscr{H}$ is called a 'Cartan preserving automorphism'. As such automorphisms are closely associated with the transformations that leave the root structure of $\tilde{\mathscr{L}}$ invariant, these transformations will be studied first. (A number (but not all) of the results of this section have been obtained previously (with different notations and conventions) by Gorman *et al* [5].)

2.1. Root-preserving transformations of $\tilde{\mathscr{L}}$

Let τ be a linear operator acting in $\mathscr{H}^*$ that is such that if α is a root of $\tilde{\mathscr{L}}$ then $\tau(\alpha)$ is also a root of $\tilde{\mathscr{L}}$. The set of these operators clearly forms a group, which will be called 'the group of root-preserving transformations of $\tilde{\mathscr{L}}$', and which will be denoted by $\mathscr{R}(\tilde{\mathscr{L}})$. Because of the root structure of $\tilde{\mathscr{L}}$ it is obvious that for the imaginary root δ

$$\tau(\delta)=\mu\delta \tag{1}$$

where

$$\mu = \pm 1. \tag{2}$$

Moreover, if α_k is the extension of the simple root α_k^0 of $\tilde{\mathscr{L}}^0$ (for $k=1,2,\ldots,l$) then

$$\tau(\alpha_k) = \sum_{j=1}^{l} (\boldsymbol{\tau}^0)_{jk}\alpha_j + n_k\delta \tag{3}$$

where $\{n_1, n_2, \ldots, n_l\}$ is a set of l integers, and where $\boldsymbol{\tau}^0$ is an $l\times l$ matrix with integer entries.

It will be convenient to rewrite (3) in an equivalent way. Defining τ^0 as a linear operator acting in $\mathscr{H}^*$ by

$$\tau^0(\alpha_k) = \sum_{j=1}^{l} (\boldsymbol{\tau}^0)_{jk}\alpha_j \tag{4}$$

(for $k=1,2,\ldots,l$), (3) can be rewritten as

$$\tau(\alpha_k) = \tau^0(\alpha_k) + n_k\delta. \tag{5}$$

Similarly there is a corresponding linear operator acting in $\mathscr{H}^{0*}$, which may also be denoted without confusion by τ^0, such that

$$\tau^0(\alpha_k^0) = \sum_{j=1}^{l} (\boldsymbol{\tau}^0)_{jk}\alpha_j^0 \tag{6}$$

(for $k=1,2,\ldots,l$). (Thus, if α is the extension of α^0, then $\tau^0(\alpha)$ is the extension of $\tau^0(\alpha^0)$.) Clearly the operator τ^0 of (6) maps every simple root of $\tilde{\mathscr{L}}^0$ into a root of $\tilde{\mathscr{L}}^0$. As the inverse operator $(\tau^0)^{-1}$ also maps every simple root of $\tilde{\mathscr{L}}^0$ into a root of $\tilde{\mathscr{L}}^0$, it follows that the matrix $(\boldsymbol{\tau}^0)^{-1}$ exists and also has integer entries. Thus the quantities κ_k^Ω defined by

$$\kappa_k^\Omega = -\sum_{j=1}^{l} ((\boldsymbol{\tau}^0)^{-1})_{jk}\mu n_j \tag{7}$$

are all integers (for $k=1,2,\ldots,l$). The 'coweight lattice' $Q_w^{0\vee}$ of $\tilde{\mathscr{L}}^0$ is defined to consist of all linear functionals Ω^0 on $\mathscr{H}^0$ that have the form

$$\Omega^0 = \sum_{k=1}^{l} \{2\kappa_k^\Omega/\langle\alpha_k^0, \alpha_k^0\rangle^0\}\Lambda_k^0 \tag{8}$$

where each κ_k^Ω (for $k=1,2,\ldots,l$) is allowed to take any integer value and Λ_k^0 are the fundamental weights of $\tilde{\mathscr{L}}^0$. (Incidentally, if $\omega^0 \in Q^{0\vee}$ (the 'scaled root lattice'), then $\omega^0 \in Q_w^{0\vee}$, but the converse proposition is not true in general.) Then, if Ω is the extension in $\mathscr{H}$ of Ω^0,

$$\langle\alpha_k, \Omega\rangle = \langle\alpha_k^0, \Omega^0\rangle^0 = \kappa_k^\Omega. \tag{9}$$

It therefore follows that

$$\mu n_k = -\langle\tau^0(\alpha_k), \Omega\rangle = -\langle\tau^0(\alpha_k^0), \Omega^0\rangle^0. \tag{10}$$

Thus (3) can be re-expressed as

$$\tau(\alpha_k) = \tau^0(\alpha_k) - \langle\tau^0(\alpha_k), \Omega\rangle\mu\delta \tag{11}$$

for $k=1,2,\ldots,l$, and hence, for any root α of $\tilde{\mathscr{L}}$ that is an extension of a root of $\tilde{L}^0$,

$$\tau(\alpha) = \tau^0(\alpha) - \langle\tau^0(\alpha), \Omega\rangle\mu\delta. \tag{12}$$

The conclusion of this argument is therefore that every element $\tau \in \mathcal{R}(\tilde{\mathcal{L}})$ depends on the three quantities, namely a 'rotation' τ^0 of the roots of $\tilde{\mathcal{L}}^0$, an extension Ω of a coweight Ω^0, and a parameter μ (where $\mu = \pm 1$). This may be expressed by writing τ as a triple in the form

$$\tau = \{\tau^0, \Omega, \mu\} \tag{13}$$

with (1) and (12) giving the basic associated root transformations. (The analysis of Gorman *et al* [5] is restricted to the special case $\mu = 1$. The appearance of the coweight lattice in this connection was noted previously by Kac and Wakimoto [6].)

It follows from (1) and (12) that the product of two such triples $\tau_1 = \{\tau_1^0, \Omega_1, \mu_1\}$ and $\tau_2 = \{\tau_2^0, \Omega_2, \mu_2\}$ is given by

$$\tau_1\tau_2 = \{\tau_1^0, \Omega_1, \mu_1\}\{\tau_2^0, \Omega_2, \mu_2\} = \{\tau_1^0\tau_2^0, \mu_2\Omega_1 + \tau_1^0(\Omega_2), \mu_1\mu_2\} \tag{14}$$

and the inverse of $\tau = \{\tau^0, \Omega, \mu\}$ is given by

$$\tau^{-1} = \{(\tau^0)^{-1}, -\mu(\tau^0)^{-1}(\Omega), \mu\}. \tag{15}$$

Thus $\tau_1 = \{\tau_1^0, \Omega_1, \mu_1\}$ and $\tau_2 = \{\tau_2^0, \Omega_2, \mu_2\}$ are conjugate via a root transformation $\phi = \{\phi^0, \Phi, \zeta\}$ (that is, $\tau_1 = \phi\tau_2\phi^{-1}$) if and only if

$$\tau_1^0 = \phi^0\tau_2^0(\phi^0)^{-1} \tag{16}$$

$$\Omega_1 = \mu_2\zeta\Phi + \zeta\phi^0(\Omega_2) - \zeta(\phi^0\tau_2^0(\phi^0)^{-1})(\Phi) \tag{17}$$

and

$$\mu_1 = \mu_2. \tag{18}$$

A root-preserving transformation τ is said to be *involutive* if τ^2 is the identity mapping 1. Thus, by (14), $\tau = \{\tau^0, \Omega, \mu\}$ is involutive if and only if

$$\tau^0(\tau^0(\alpha^0)) = \alpha^0 \tag{19}$$

for every $\alpha^0 \in \Delta^0$ and

$$\tau^0(\Omega) = -\mu\Omega. \tag{20}$$

In (1) and (12) the effect of τ has been confined to the subspace of $\mathcal{H}^*$ that is spanned by the roots $\alpha_0, \alpha_1, \ldots, \alpha_l$. However, it follows from the analysis of the next subsection that τ can be extended to act on the remaining basis element Λ_0 of $\mathcal{H}^*$ to give

$$\tau(\Lambda_0) = \mu\Lambda_0 + \langle\Lambda_0, \delta\rangle\{\Omega - \tfrac{1}{2}\langle\Omega, \delta\rangle\mu\delta\}. \tag{21}$$

It is easily shown that for any two roots α and β of $\tilde{\mathcal{L}}$

$$\langle\tau(\alpha), \tau(\beta)\rangle = \langle\alpha, \beta\rangle \tag{22}$$

and that for any two roots α and β of $\tilde{\mathcal{L}}$ that are extensions of roots of $\tilde{\mathcal{L}}^0$

$$\langle\tau^0(\alpha), \tau^0(\beta)\rangle = \langle\alpha, \beta\rangle. \tag{23}$$

For each ω that is an extension of a linear functional $\omega^0 \in Q^{0\vee}$ a linear operator T_ω acting on $\mathcal{H}^*$ may be defined by

$$T_\omega(\lambda) = \lambda + \langle\lambda, \delta\rangle\omega - \{\langle\lambda, \omega\rangle + \tfrac{1}{2}\langle\omega, \omega\rangle\langle\lambda, \delta\rangle\}\delta \tag{24}$$

for every linear functional λ defined on $\mathcal{H}$. Then $T_\omega T_{\omega'} = T_{\omega+\omega'}$ for all such ω and ω'. It is well known that the set of such 'translations' (which may be denoted by $\mathcal{T}$) forms an Abelian invariant subgroup of the Weyl group $\mathcal{W}$ of $\tilde{\mathcal{L}}$, and that $\mathcal{W}$ is the semidirect product of $\mathcal{T}$ and $\mathcal{W}^0$, where $\mathcal{W}^0$ is the subgroup of $\mathcal{W}$ consisting of elements S^0 that are generated by the Weyl reflections S_{α_k} for $k = 1, 2, \ldots, l$ (so that $\mathcal{W}^0$ is isomorphic to the Weyl group of $\tilde{\mathcal{L}}^0$).

As every element of the Weyl group $\mathcal{W}$ of $\tilde{\mathcal{L}}$ maps roots of $\tilde{\mathcal{L}}$ into roots of $\tilde{\mathcal{L}}$, $\mathcal{W}$ must be a subgroup of $\mathcal{R}(\tilde{\mathcal{L}})$. Comparison of (24) and the standard definition of S_α with (21), (12) and (1) shows that the Weyl group element that is the product of the element s^0 of $\mathcal{W}^0$ and the translation T_ω of $\mathcal{T}$ corresponds to a triple with $\tau^0 = S^0$, $\Omega = \omega$ and $\mu = 1$. However, $\mathcal{W}$ is a *proper* subgroup of $\mathcal{R}(\tilde{\mathcal{L}})$, for in general $\mathcal{R}(\tilde{\mathcal{L}})$ contains elements that do not belong to $\mathcal{W}$ by virtue of possessing one or more of the following three features:

(i) $\mu = -1$;

(ii) Ω is the extension of a linear functional Ω^0 that is not a member of the sublattice $Q^{0\vee}$;

(iii) τ^0 corresponds to a symmetry of the Dynkin diagram of $\tilde{\mathcal{L}}^0$.

2.2. Cartan preserving automorphisms of $\tilde{\mathcal{L}}$ corresponding to root-preserving transformations

Every Cartan preserving automorphism ψ of $\tilde{\mathcal{L}}$ induces a root-preserving transformation of $\tilde{\mathcal{L}}$, for applying ψ to both sides of $[h, a_\alpha] = \alpha(h)a_\alpha$ and putting $h' = \psi(h)$ gives

$$[h', \psi(a_\alpha)] = \alpha(\psi^{-1}(h'))\psi(a_\alpha). \tag{25}$$

Let τ be a root-preserving transformation of $\tilde{\mathcal{L}}$ and let ψ_τ be any Cartan preserving automorphism that corresponds to τ, so that, by (25),

$$\tau(\alpha)(h) = \alpha(\psi_\tau^{-1}(h)) \tag{26}$$

for all $h \in \mathcal{H}$ and every root α of $\tilde{\mathcal{L}}$. It then follows that

$$B(\psi_\tau(a), \psi_\tau(b)) = B(a, b) \tag{27}$$

for all $a, b \in \tilde{\mathcal{L}}$, and hence that

$$\psi_\tau(h_\alpha) = h_{\tau(\alpha)} \tag{28}$$

for every $\alpha \in \Delta$. Thus, in particular, as $h_\delta = c$, by (28) and (1),

$$\psi_\tau(c) = \mu c \tag{29}$$

and as $h_{\alpha_k} = t^0 \otimes h^0_{\alpha^0_k}$, by (3),

$$\psi_\tau(t^0 \otimes h^0_{\alpha^0_k}) = \sum_{j=1}^{l} (\tau^0)_{jk} t^0 \otimes h^0_{\alpha^0_j} + n_k \mu c \tag{30}$$

for $k = 1, 2, \ldots, l$. Equivalently, by virtue of (12) and (10),

$$\psi_\tau(t^0 \otimes h^0_{\alpha^0}) = t^0 \otimes h^0_{\tau^0(\alpha^0)} - \langle \tau^0(\alpha^0), \Omega^0 \rangle^0 \mu c \tag{31}$$

for every root α^0 of $\tilde{\mathcal{L}}^0$. It should be noted that the right-hand sides of (29)-(31) do not contain any terms involving the scaling element d.

Turning to the effect of automorphisms on d, as the only appearance of d in the basic commutation relations

$$[t^j \otimes a^0, t^k \otimes b^0] = t^{j+k} \otimes [a^0, b^0] + j\delta^{j+k,0} B^0(a^0, b^0)c \tag{32}$$

(for all integers j and k and all $a^0, b^0 \in \tilde{\mathscr{L}}^0$),

$$[t^j \otimes a^0, c] = 0 \tag{33}$$

(for all integers j and all $a^0 \in \tilde{\mathscr{L}}^0$),

$$[d, t^j \otimes a^0] = jt^j \otimes a^0 \tag{34}$$

(for all integers j and all $a^0 \in \tilde{\mathscr{L}}^0$), and

$$[d, c] = 0 \tag{35}$$

is in the left-hand sides of (34) and (35), it is obvious there exists a set of automorphisms ϕ_ξ of $\tilde{\mathscr{L}}$ such that ϕ_ξ leaves invariant every basis element except d and

$$\phi_\xi(d) = d + \xi c \tag{36}$$

where ξ is *any* complex number. Clearly these automorphisms ϕ_ξ form an Abelian subgroup of the group of Cartan preserving automorphisms of $\tilde{\mathscr{L}}$ and each one commutes with *every* automorphism of this group. Thus the group of Cartan preserving automorphisms of $\tilde{\mathscr{L}}$ is the direct product of this Abelian subgroup with a non-trivial subgroup, and henceforth attention will be concentrated on this non-trivial subgroup. Applying the automorphism ψ_τ to (34) and using (29) and (31) gives

$$\psi_\tau(d) = \mu d + h_\Omega + \eta c \tag{37}$$

where η is any complex number. As the arbitrariness of η can be absorbed in the automorphism ϕ_ξ of (36), one may make any particular choice of value for the remaining automorphisms, of which the most convenient is

$$\eta = -\tfrac{1}{2}\langle\Omega, \Omega\rangle\mu = -\tfrac{1}{2}\langle\Omega^0, \Omega^0\rangle^0\mu \tag{38}$$

and with this choice (37) gives

$$\psi_\tau(d) = \mu d + h_\Omega - \tfrac{1}{2}\langle\Omega^0, \Omega^0\rangle^0 \mu c. \tag{39}$$

(One motivation for the choice (38) is that the relation (21) then follows from (39) on assuming that (26) can be extended to apply to any linear functional α on $\mathscr{H}$, and in particular to Λ_0, and, as noted previously, (21) is consistent with the effect of the Weyl group operations.)

It only remains to consider the effect of ψ_τ on the basis elements of the root subspaces of $\tilde{\mathscr{L}}$. For any root α of $\tilde{\mathscr{L}}$

$$\psi_\tau(e_\alpha) = \chi_\alpha \, e_{\tau(\alpha)} \tag{40}$$

where χ_α is some complex number. As each *real* root of $\tilde{\mathscr{L}}$ has the form $j\delta + \alpha$, where α is the extension of a root α^0 of $\tilde{\mathscr{L}}^0$ and j is an integer, and as the root subspace $\tilde{\mathscr{L}}_{j\delta+\alpha}$ is one-dimensional, this implies that

$$\psi_\tau(t^j \otimes e^0_{\alpha^0}) = \chi_{j\delta+\alpha} \, t^{(j - \langle\tau^0(\alpha^0), \Omega^0\rangle^0)\mu} \otimes e^0_{\tau^0(\alpha^0)}. \tag{41}$$

Similarly, as each *imaginary* root of $\tilde{\mathscr{L}}$ has the form $j\delta$, where j is an integer, and as the root subspace $\tilde{\mathscr{L}}_{j\delta}$ is l-dimensional,

$$\psi_\tau(t^j \otimes h^0_{\alpha^0_k}) = \chi_{j\delta} t^j \otimes h^0_{\tau^0(\alpha^0_k)} \tag{42}$$

for $k=1,2,\ldots,l$, where the complex number $\chi_{j\delta}$ is independent of k. In (40)–(42) $\chi_{j\delta+\alpha}$ and $\chi_{j\delta}$ are such that

$$\chi_{j\delta+\alpha}=\chi_{j\delta}\chi_{\alpha}. \tag{43}$$

It is easily shown that

$$\chi_{\alpha+\beta}=\{N^{0}_{\tau^0(\alpha^0),\tau^0(\beta^0)}/N^{0}_{\alpha^0,\beta^0}\}\chi_\alpha\chi_\beta \tag{44}$$

where α and β are extensions of α^0 and β^0 respectively, and $\alpha^0+\beta^0$ is a non-zero root of $\tilde{\mathscr{L}}^0$, and where the structure constants $N^0_{\alpha^0,\beta^0}$ of $\tilde{\mathscr{L}}^0$ are defined in the usual way by

$$[e^0_{\alpha^0},e^0_{\beta^0}]=N^0_{\alpha^0,\beta^0}\,e^0_{\alpha^0+\beta^0}. \tag{45}$$

Also

$$\chi_{(j+j')\delta}=\chi_{j\delta}\chi_{j'\delta} \tag{46}$$

for all integers j and j' such that $j+j'\neq 0$.

For certain $b\in\tilde{\mathscr{L}}$ an operator $\exp(\mathrm{ad}(b))$ may be defined on each $a\in\tilde{\mathscr{L}}$ by

$$\exp(\mathrm{ad}(b))(a)=\sum_{n=0}^{\infty}\{1/n!\}\{\mathrm{ad}(b)\}^n(a) \tag{47}$$

where $\mathrm{ad}(b)$ is defined in the usual way by

$$\mathrm{ad}(b)(a)=[b,a] \tag{48}$$

for all $a\in\tilde{\mathscr{L}}$, provided that the series on the right-hand side of (47) converges.

The most important examples are those with $b=h'$, where $h'\in\mathscr{H}$. In this case

$$\exp(\mathrm{ad}(h'))(h)=h \tag{49}$$

for all $h\in\mathscr{H}$ and

$$\exp(\mathrm{ad}(h'))(a_\alpha)=\{\exp\alpha(h')\}a_\alpha \tag{50}$$

for any root $\alpha\in\Delta$. Clearly the operator

$$\Theta_{h'}=\exp(\mathrm{ad}(h')) \tag{51}$$

is a Cartan preserving automorphism of $\tilde{\mathscr{L}}$ for which

$$\tau^0=1 \qquad \Omega=0 \qquad \mu=1 \qquad \text{and} \qquad \chi_\alpha=\exp\alpha(h') \qquad \forall\alpha\in\Delta. \tag{52}$$

Another interesting automorphism of $\tilde{\mathscr{L}}$ involving operators of the type defined in (47) is

$$\exp(\mathrm{ad}(-E_{\alpha_0}))\exp(\mathrm{ad}(E_{-\alpha_0}))\exp(\mathrm{ad}(-E_{\alpha_0})) \tag{53}$$

which, as noted by Frenkel and Kac [7], corresponds to the Weyl reflection S_{α_0}.

The above considerations imply that the most general Cartan preserving automorphism of $\tilde{L}$ is a product of automorphisms of the following three types:

(i) ψ_τ (as specified in (29), (31), (39)–(42)) for some root-preserving transformation $\tau=\{\tau^0,\Omega,\mu\}$, with $\chi_{\alpha_k}=1$ for $k=1,2,\ldots,l$, $\chi_\delta=1$, and $\chi_{-\alpha}=\chi_\alpha$ for all $\alpha\in\delta$;

(ii) $\Theta_{h'}$ (as defined in (51)) for some $h'\in\mathscr{H}$;

(iii) ϕ_ξ (as defined in (36)) for some complex number ξ.

In the analysis of the succeeding sections one of the aims is to find a representative of each conjugacy class of the group of automorphisms of $\tilde{\mathscr{L}}$, and for this the following results are very useful:

(i) Every conjugacy class of the group of automorphisms of $\tilde{\mathscr{L}}$ contains a Cartan preserving automorphism of $\tilde{\mathscr{L}}$ (for a proof see Levstein [8]).

(ii) For the Cartan preserving automorphisms ψ_τ and $\exp(\mathrm{ad}(h'))$ (with $h' \in \mathscr{H}$)

$$\exp\{\mathrm{ad}(h')\}\psi_\tau = \psi_\tau \exp\{\mathrm{ad}(\psi_\tau^{-1}(h'))\}. \tag{54}$$

(iii) Consider the automorphism $\psi_\tau \exp(\mathrm{ad}(h'))$, where $h' \in \mathscr{H}$. Let $\mathscr{H}_\tau^+$ and $\mathscr{H}_\tau^-$ be the subspaces of $\mathscr{H}$ that are such that $\psi_\tau(h) = h$ for $h \in \mathscr{H}_\tau^+$ and $\psi_\tau(h) = -h$ for $h \in \mathscr{H}_\tau^-$. Then, for any $h \in \mathscr{H}$, $h = h^+ + h^-$, where $h^+ \in \mathscr{H}_\tau^+$ and $h = \in \mathscr{H}_\tau^-$. As (54) implies that

$$\exp\{\mathrm{ad}(\tfrac{1}{2}h^-)\}\psi_\tau \exp\{\mathrm{ad}(h^+ + h^-)\}(\exp\{\mathrm{ad}(\tfrac{1}{2}h^-)\})^{-1} = \psi_\tau \exp\{\mathrm{ad}(h^+)\}$$

it follows that $\psi_\tau \exp\{\mathrm{ad}(h^+ + h^-)\}$ and $\psi_\tau \exp\{\mathrm{ad}(h^+)\}$ are conjugate. Thus, in considering the conjugacy class containing $\psi_\tau \exp(\mathrm{ad}(h'))$, it may be assumed without loss of generality that

$$\psi_\tau(h') = h'. \tag{55}$$

(iv) If ψ_τ, $\psi_{\tau'}$ and ψ_ϕ are three Cartan preserving automorphisms of $\tilde{\mathscr{L}}$ that are such that

$$\psi_{\tau'} = \psi_\phi \psi_\tau (\psi_\phi)^{-1} \tag{56}$$

then (54) implies that

$$\psi_{\tau'} \exp\{\mathrm{ad}(h')\} = \psi_\phi[\psi_\tau \exp\{\mathrm{ad}(h)\}](\psi_\phi)^{-1} \tag{57}$$

where $h \in \mathscr{H}$ and $h' = \psi_\phi(h)$. Thus $\psi_\tau \exp\{\mathrm{ad}(h)\}$ (for $h \in \mathscr{H}$) is conjugate to $\psi_{\tau'} \exp\{\mathrm{ad}(h')\}$, where h' is also a member of $\mathscr{H}$. (It should be noted that if ψ_τ, $\psi_{\tau'}$, and ρ are three automorphisms of $\tilde{\mathscr{L}}$ related by

$$\psi_{\tau'} = \rho\psi_\tau(\rho)^{-1} \tag{58}$$

and that ψ_τ and $\psi_{\tau'}$ are Cartan preserving automorphisms, but ρ is not a Cartan preserving automorphism, then these conclusions do not apply, because $h' = \rho(h)$ is not a member of $\mathscr{H}$.)

(v) A necessary condition for (56) to hold is that the corresponding root transformations τ and τ' must conjugate via the root transformation ϕ, so that (16)–(18) must apply (with $\tau_1 = \tau$ and $\tau_2 = \tau'$).

(vi) If $\psi_\tau \exp\{\mathrm{ad}(h')\}$ is involutive then ψ_τ must be involutive. A necessary condition for ψ_τ to be involutive is that the corresponding root transformation τ must be involutive, and so (19) and (20) must be satisfied. Moreover, by (54) and (55), $\exp\{\mathrm{ad}(h')\}$ must also be involutive, which requires that h' must be such that

$$\exp\{\alpha_k(h') = \pm 1 \tag{59}$$

for all $k = 0, 1, 2, \ldots, l$.

Unfortunately the situation mentioned in connection with (58), where ψ_τ and $\psi_{\tau'}$ are Cartan preserving automorphisms, but ρ is not a Cartan preserving automorphism, is quite common. (This situation also occurs in the case of simple Lie algebras, for it was shown by Gantmacher [2] that every inner automorphism of a simple Lie algebra (including those associated with the Weyl reflections) is conjugate to a 'chief inner automorphism' of the form $\exp\{\mathrm{ad}(h')\}$ (for some h' of its Cartan subalgebra). However, a chief inner automorphism cannot be conjugate to an automorphism associated with a Weyl reflection via a Cartan preserving automorphism (as in (56)), because this requires that the corresponding root transformations must be conjugate, which is impossible as $\exp\{\mathrm{ad}(h')\}$ corresponds to the identity root transformation.) It is therefore essential to have the means of dealing with the most general automorphisms of $\tilde{\mathscr{L}}$. These will be considered in the next section.

3. The matrix formulation of general automorphisms of $\tilde{\mathscr{L}}$

3.1. $\tilde{\mathscr{L}}$ in matrix form

Let $\mathbf{\Gamma}$ be a faithful irreducible representation of some dimension d_Γ of the simple Lie algebra $\tilde{\mathscr{L}}^0$. Then the first term of the general element $\Sigma_{j=-\infty}^{\infty} \Sigma_{p=1}^{n^0} \mu_{jp} t^j \otimes a_p^0 + \mu_c c + \mu_d d$ of the affine untwisted Kac-Moody algebra $\tilde{\mathscr{L}}$ is represented by the $d_\Gamma \times d_\Gamma$ matrix $\Sigma_{j=-\infty}^{\infty} \Sigma_{p=1}^{n^0} \mu_{jp} t^j \mathbf{\Gamma}(a_p^0)$. A typical matrix of this form will be denoted by $\boldsymbol{a}(t)$, i.e.

$$\boldsymbol{a}(t) = \sum_{j=-\infty}^{\infty} \sum_{p=1}^{n^0} \mu_{jp} t^j \mathbf{\Gamma}(a_p^0). \tag{60}$$

Clearly all the entries of $\boldsymbol{a}(t)$ are Laurent polynomials in the complex variable t. A typical element of $\tilde{\mathscr{L}}$ can then be written as

$$\boldsymbol{a}(t) + \mu_c c + \mu_d d. \tag{61}$$

(Of course in *no* sense do the + signs in (61) represent ordinary matrix addition.)

It follows from (32) that the commutator of two matrices of the form (60) is given by

$$[\boldsymbol{a}(t), \boldsymbol{b}(t)] = \boldsymbol{a}(t)\boldsymbol{b}(t) - \boldsymbol{b}(t)\boldsymbol{a}(t) + \psi(\boldsymbol{a}(t), \boldsymbol{b}(t))c \tag{62}$$

where

$$\psi(\boldsymbol{a}(t), \boldsymbol{b}(t)) = \frac{1}{\gamma} \operatorname{Res}\left\{\operatorname{tr}\left(\frac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t} \boldsymbol{b}(t)\right)\right\}. \tag{63}$$

Here $\operatorname{Res}\{f(t)\}$ denotes the residue of the function $f(t)$ at $t = 0$ (so that $\operatorname{Res}\{t^{j-1}\} = \delta^{j,0}$), and γ is the Dynkin index of the representation $\mathbf{\Gamma}$ of the simple Lie algebra $\tilde{\mathscr{L}}^0$, which is such that

$$\operatorname{tr}\{\mathbf{\Gamma}(a^0)\mathbf{\Gamma}(b^0)\} = \gamma B^0(a^0, b^0) \tag{64}$$

for all $a^0, b^0 \in \tilde{\mathscr{L}}^0$. The commutation relation (33) implies that

$$[c, \boldsymbol{a}(t)] = 0 \tag{65}$$

while (34) implies that

$$[d, \boldsymbol{a}(t)] = t \frac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t} \tag{66}$$

and (35) is unchanged.

3.2. The four matrix types of automorphism of $\tilde{\mathscr{L}}$

It will first be shown that in the matrix formulation there exist four types of automorphism of a complex untwisted affine Kac-Moody algebra $\tilde{\mathscr{L}}$, which will be referred to as type 1a, type 1b, type 2a and type 2b.

Each type of atuomorphism depends on the following three quantities (although the dependence is different for the different types):

(i) a $d_\Gamma \times d_\Gamma$ matrix $U(t)$, which is assumed to be invertible and for which all the entries of $U(t)$ and $U(t)^{-1}$ are assumed to be Laurent polynomials in the complex variable t;

(ii) a non-zero complex parameter u;

(iii) an arbitrary complex parameter ξ.

It is convenient to exhibit these together as a triple in the form $\{U(t), u, \xi\}$. The explicit actions of the corresponding automorphisms ϕ are as follows:

(i) Actions on $\boldsymbol{a}(t)$:

(a) For type 1a automorphisms:

$$\phi(\mathbf{a}(t)) = \boldsymbol{U}(t)\boldsymbol{a}(ut)\boldsymbol{U}(t)^{-1} + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(\boldsymbol{U}(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}\boldsymbol{a}(ut)\right)\right\}c. \tag{67}$$

(b) For type 1b automorphisms:

$$\phi(\boldsymbol{a}(t)) = \boldsymbol{U}(t)\{-\tilde{\boldsymbol{a}}(ut)\}\boldsymbol{U}(t)^{-1} + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(\boldsymbol{U}(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}(-\tilde{\boldsymbol{a}}(ut))\right)\right\}c. \tag{68}$$

(c) For type 2a automorphisms:

$$\phi(\boldsymbol{a}(t)) = \boldsymbol{U}(t)\boldsymbol{a}(ut^{-1})\boldsymbol{U}(t)^{-1} + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(\boldsymbol{U}(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}\boldsymbol{a}(ut^{-1})\right)\right\}c. \tag{69}$$

(d) For type 2b automorphisms:

$$\phi(\boldsymbol{a}(t)) = \boldsymbol{U}(t)\{-\tilde{\boldsymbol{a}}(ut^{-1})\}\boldsymbol{U}(t)^{-1} + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(\boldsymbol{U}(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}(-\tilde{\boldsymbol{a}}(ut^{-1}))\right)\right\}c. \tag{70}$$

(ii) Actions on c:

$$\phi(c) = \mu c \tag{71}$$

where

$$\mu = \begin{cases} 1 & \text{for type 1a and 1b} \\ -1 & \text{for type 2a and 2b.} \end{cases} \tag{72}$$

(iii) Actions on d:

$$\phi(d) = \mu\boldsymbol{\Phi}(\boldsymbol{U}(t)) + \xi c + \mu d \tag{73}$$

where $\boldsymbol{\Phi}(\boldsymbol{U}(t))$ is the $d_\Gamma \times d_\Gamma$ matrix that depends on $\boldsymbol{U}(t)$ according to the definition

$$\boldsymbol{\Phi}(\boldsymbol{U}(t)) = \left\{-t\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}\boldsymbol{U}(t)^{-1} + \frac{1}{d_\Gamma}\operatorname{tr}\left(t\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}\boldsymbol{U}(t)^{-1}\right)\mathbf{1}\right\} \tag{74}$$

and μ is defined in (72).

It should be noted that the μ and ξ of (71) and (73) may be identified with the corresponding quantities of section 2. (It is implicitly assumed here that the first terms on the right-hand sides of (67)–(70) are of the form $\Sigma_{j=-\infty}^{\infty}\Sigma_{p=1}^{n^0}\mu'_{jp}t^j\boldsymbol{\Gamma}(a_p^0)$ for some some complex numbers μ'_{jp}, so that the corresponding mappings ϕ are indeed mappings of $\tilde{\mathscr{L}}$ onto $\tilde{\mathscr{L}}$.)

One particular mapping that is well worthy of note is the 'Cartan involution' ϕ_{Cartan}, which, with the conventions of [4], is the involutive automorphism of $\tilde{\mathscr{L}}$ that is defined by

$$\phi_{\text{Cartan}}(h) = -h \tag{75}$$

for all $h \in \mathscr{H}$, and

$$\phi_{\text{Cartan}}(e_\alpha) = e_{-\alpha} \tag{76}$$

for all $\alpha \in \Delta$. With the assumptions that $\Gamma(h^0_{\alpha^0})$ is diagonal and that $\Gamma(e^0_{\alpha^0})$ is real for all $\alpha^0 \in \Delta^0$, it is easily checked that ϕ_{Cartan} is a type 2b automorphism with $\boldsymbol{U}(t) = \mathbf{1}$, $u = 1$ and $\xi = 0$.

The rest of this subsection will be devoted the motivation for the considering mappings of these types, and to a demonstration (in outline) that they do indeed provide structure-preserving mappings of $\tilde{\mathscr{L}}$ into $\tilde{\mathscr{L}}$. For convenience of exposition this demonstration will be restricted to the case of type 1a automorphisms, the arguments for the other types being similar.

Consider first a mapping ϕ such that

$$\phi(\boldsymbol{a}(t)) = \boldsymbol{U}(t)\boldsymbol{a}(t)\boldsymbol{U}(t)^{-1} + \phi_c(\boldsymbol{a}(t))c + \phi_d(\boldsymbol{a}(t))d \tag{77}$$

where $\phi_c(\boldsymbol{a}(t))$ and $\phi_d(\boldsymbol{a}(t))$ are complex numbers that depend on $\boldsymbol{a}(t)$,

$$\phi(c) = \mu c \tag{78}$$

where $\mu = \pm 1$, and

$$\phi(d) = \boldsymbol{\phi}(t) + \xi c + \lambda d \tag{79}$$

where $\boldsymbol{\phi}(t)$ is a $d_\Gamma \times d_\Gamma$ matrix whose elements are Laurent polynomials in t and where ξ and λ are complex numbers.

The automorphism condition

$$[\phi(\boldsymbol{a}(t)), \phi(\boldsymbol{b}(t))] = \phi([\boldsymbol{a}(t), \boldsymbol{b}(t)]) \tag{80}$$

for all $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ of $\tilde{\mathscr{L}}$ implies that

$$\psi(\boldsymbol{U}(t)\boldsymbol{a}(t)\boldsymbol{U}(t)^{-1}, \boldsymbol{U}(t)\boldsymbol{b}(t)\boldsymbol{U}(t)^{-1})c - \psi(\boldsymbol{a}(t), \boldsymbol{b}(t))\mu c \tag{81}$$

$$-\phi_d\left(\boldsymbol{b}(t)\frac{\mathrm{d}}{\mathrm{d}t}\{\boldsymbol{U}(t)\boldsymbol{a}(t)\boldsymbol{U}(t)^{-1}\}\right)$$

$$+\phi_d\left(\boldsymbol{a}(t)\frac{\mathrm{d}}{\mathrm{d}t}\{\boldsymbol{U}(t)\boldsymbol{b}(t)\boldsymbol{U}(t)^{-1}\}\right) \tag{82}$$

$$= \phi_c(\boldsymbol{a}(t)\boldsymbol{b}(t) - \boldsymbol{b}(t)\boldsymbol{a}(t))c + \phi_d(\boldsymbol{a}(t)\boldsymbol{b}(t) - \boldsymbol{b}(t)\boldsymbol{a}(t))d. \tag{83}$$

Equating the coefficients of d in (83) gives

$$\phi_d(\boldsymbol{a}(t)\boldsymbol{b}(t) - \boldsymbol{b}(t)\boldsymbol{a}(t)) = 0$$

for all $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ of $\tilde{\mathscr{L}}$, and as $\tilde{\mathscr{L}}^0$ is simple, so that every element of $\tilde{\mathscr{L}}^0$ can be written in the form $[a^0, b^0]$ for some $a^0, b^0 \in \tilde{\mathscr{L}}^0$, it follows that

$$\phi_d(\boldsymbol{a}(t)) = 0 \tag{84}$$

for all $\boldsymbol{a}(t)$ of $\tilde{\mathscr{L}}$. Equating the coefficients of c in (83) and using (63) then gives

$$\phi_c(\boldsymbol{a}(t)\boldsymbol{b}(t) - \boldsymbol{b}(t)\boldsymbol{a}(t))$$

$$= \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(\boldsymbol{U}(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}\{\boldsymbol{a}(t)\boldsymbol{b}(t) - \boldsymbol{b}(t)\boldsymbol{a}(t)\}\right)\right\}$$

$$+\frac{1-\mu}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(\frac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t}\boldsymbol{b}(t)\right)\right\} \tag{85}$$

for all $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ of $\tilde{\mathscr{L}}$. However, this is consistent only if $\mu=1$, for, on letting $\boldsymbol{a}(t)=t^j\boldsymbol{\Gamma}(a^0)$ and $\boldsymbol{b}(t)=t^{-j}\boldsymbol{\Gamma}(b^0)$ for any $a^0, b^0\in\tilde{\mathscr{L}}^0$ and for any integer j, the left-hand side of (85) is j-independent, whereas the right-hand side of (85) is j-dependent unless $\mu=1$. On taking

$$\mu=1 \tag{86}$$

the second term on the right-hand side of (85) vanishes, and as $\tilde{\mathscr{L}}^0$ is simple, it follows that

$$\phi_c(\boldsymbol{a}(t))=\frac{1}{\gamma}\,\mathrm{Res}\left\{\mathrm{tr}\left(\boldsymbol{U}(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}\,\boldsymbol{a}(t)\right)\right\}. \tag{87}$$

Substituting (84), (86) and (87) into (77) and (78) then gives (67) (with $u=1$) and (71) (with $\mu=1$).

Similarly substituting (77) and (78) into the automorphism condition

$$[\phi(d),\phi(\boldsymbol{a}(t))]=\phi([d,\boldsymbol{a}(t)]) \tag{88}$$

(for all $\boldsymbol{a}(t)$ of $\tilde{\mathscr{L}}$), and equating coefficients of c gives

$$\psi(\boldsymbol{\phi},\boldsymbol{U}(t)\boldsymbol{a}(t)\boldsymbol{U}(t)^{-1})=\phi_c\left(t\frac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t}\right) \tag{89}$$

while consideration of the 'matrix part' gives

$$\boldsymbol{v}(t)\boldsymbol{a}(t)-\boldsymbol{a}(t)\boldsymbol{v}(t)=(1-\lambda)t\frac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t} \tag{90}$$

for all $\boldsymbol{a}(t)$ of $\tilde{\mathscr{L}}$, where

$$\boldsymbol{v}(t)=\boldsymbol{U}(t)^{-1}\boldsymbol{\phi}\boldsymbol{U}(t)+\lambda t\boldsymbol{U}(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}. \tag{91}$$

This has to hold for all $\boldsymbol{a}(t)$. In particular, for $\boldsymbol{a}(t)=\boldsymbol{\Gamma}(a^0)$, where a^0 is any element of $\tilde{\mathscr{L}}^0$, i.e. for $\boldsymbol{a}(t)$ independent of t, (90) reduces to $\boldsymbol{v}(t)\boldsymbol{\Gamma}(a^0)-\boldsymbol{\Gamma}(a^0)\boldsymbol{v}(t)=0$. Thus, by Schur's lemma, $\boldsymbol{v}(t)=\kappa(t)\boldsymbol{1}$, for some Laurent polynomial $\kappa(t)$. Then (90) with $\boldsymbol{a}(t)$ dependent on t implies that

$$\lambda=1 \tag{92}$$

and hence, by (91),

$$\boldsymbol{\phi}=-t\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}\,\boldsymbol{U}(t)^{-1}+\kappa(t)\boldsymbol{1}. \tag{93}$$

It is then easy to check that (89) is satisfied by this expression (65) for $\boldsymbol{\phi}$. Finally, as $\boldsymbol{\Gamma}(a^0)$ has zero trace for all $a^0\in\tilde{\mathscr{L}}^0$, it follows that $\mathrm{tr}(\boldsymbol{\phi})=0$, and so (93) implies that

$$\kappa(t)=\frac{1}{d_\Gamma}\,\mathrm{tr}\left(t\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}\,\boldsymbol{U}(t)^{-1}\right) \tag{94}$$

which gives (73) (with $\mu=1$).

The foregoing argument has established that the mapping ϕ defined in (67) with $u=1$ and in (71) and (73) (with $\mu=1$) is a structure-preserving mapping of $\tilde{\mathscr{L}}$ into $\tilde{\mathscr{L}}$. Clearly the inner automorphisms of the subalgebra of $\tilde{\mathscr{L}}$ that is isomorphic to $\tilde{\mathscr{L}}^0$ are contained in the special case in which $\boldsymbol{U}(t)$ is independent of t (and $\xi=0$). In particular these include all the mappings of the form $\exp\{\mathrm{ad}(h')\}$ with $h' = t^0 \otimes h'^0$, where $h'^0 \in \mathcal{H}^0$. By virtue of (33) and (35) the mapping $\exp\{\mathrm{ad}(c)\}$ is just the identity mapping, and so needs no further consideration. However, mappings of the form $\exp\{\mathrm{ad}(d)\}$ are not included in this special case, and it will now be shown that to include them it is necessary to introduce the arbitrary non-zero complex factor u in (67).

It follows from the definition (47) and from (34) that for any complex number λ

$$\exp\{\mathrm{ad}(\lambda d)\}(t^j \otimes a^0) = \mathrm{e}^{\lambda j}(t^j \otimes a^0) \tag{95}$$

where j is any integer and a^0 is any element of $\tilde{\mathcal{L}}^0$. Similarly, from (35),

$$\exp\{\mathrm{ad}(\lambda d)\}c = c \tag{96}$$

and it is trivially true that

$$\exp\{\mathrm{ad}(\lambda d)\}d = d. \tag{97}$$

Equation (95) can be recast in matrix form as

$$\exp\{\mathrm{ad}(\lambda d)\}\boldsymbol{a}(t) = \boldsymbol{a}(\mathrm{e}^{\lambda} t). \tag{98}$$

(In particular, if $\boldsymbol{a}(t) = t^j \boldsymbol{\Gamma}(a^0)$, the right-hand side of (98) reduces to $(\mathrm{e}^{\lambda} t)^j \boldsymbol{\Gamma}(a^0)$, which is equal to $\mathrm{e}^{\lambda j}\boldsymbol{a}(t)$, and so is in accordance with (95), and this agreement obviously extends to linear combinations of expressions of the form $t^j \boldsymbol{\Gamma}(a^0)$.) Then, with u defined by

$$u = \mathrm{e}^{\lambda} \tag{99}$$

(98) becomes

$$\exp\{\mathrm{ad}(\lambda d)\}\boldsymbol{a}(t) = \boldsymbol{a}(ut). \tag{100}$$

The composition of the type 1a automorphism specified in (67) with $u = 1$ and $\exp\{\mathrm{ad}(\lambda d)\}$ acting on $(\boldsymbol{a}(t))$ is then

$$\begin{aligned}(\phi \circ \exp\{\mathrm{ad}(\lambda d)\})(\boldsymbol{a}(t)) &\\ &= \phi(\boldsymbol{a}(ut)) \\ &= \boldsymbol{U}(t)\boldsymbol{a}(ut)\boldsymbol{U}(t)^{-1} + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(\boldsymbol{U}(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}\boldsymbol{a}(ut)\right)\right\}c\end{aligned} \tag{101}$$

which gives (67) for *arbitrary* non-zero values of u.

Having established the necessity for including the factor u, it is easy to repeat the argument given above, starting with

$$\phi(\boldsymbol{a}(t)) = \boldsymbol{U}(t)\boldsymbol{a}(ut)\boldsymbol{U}(t)^{-1} + \phi_c(\boldsymbol{a}(t))c + \phi_d(\boldsymbol{a}(t))d \tag{102}$$

in place of (77), but with (78) and (79) unchanged in form, to show that (67), (71) and (73) (with $\mu = 1$) is the resulting structure-preserving mapping of $\tilde{\mathcal{L}}$. The general line of argument for the type 1b, 2a and 2b automorphisms is similar, although some of the detail is different.

This subsection will be concluded with a brief statement of how these results are related to those of Levstein [8]. In this paper on Kac–Moody algebra automorphisms, Levstein briefly mentioned two types of matrix automorphisms, which, when rewritten in the notation of the present paper (and corrected for typographical errors) are:

$$\phi(\boldsymbol{a}(t)) = \boldsymbol{U}(t)\boldsymbol{a}(t)\boldsymbol{U}(t)^{-1} + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(\boldsymbol{U}(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t}\boldsymbol{a}(t)\right)\right\}c \tag{103}$$

and

$$\phi(a(t)) = U(t)\{-\tilde{a}(t)\}U(t)^{-1} + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U(t)^{-1}\frac{\mathrm{d}U(t)}{\mathrm{d}t}(-\tilde{a}(t))\right)\right\}c. \tag{104}$$

No expressions are quoted by Levstein for $\phi(d)$, and it appears to be implicitly assumed that $\phi(c) = c$. Consequently Levstein's matrix automorphisms are essentially only automorphisms of the 'derived' algebra of $\tilde{\mathscr{L}}$, that is, of the subalgebra of $\tilde{\mathscr{L}}$ with the basis element d removed. As will be seen from the analysis given above, the inclusion of d has two consequences. Firstly, it is necessary to evaluate $\phi(d)$, that is, the effect of the corresponding automorphisms on d, and secondly, it is necessary to include the automorphisms $\exp\{\operatorname{ad}(d)\}$ in the matrix formulation (which, as shown above, is accomplished by introducing the non-zero complex number u in various terms). One further feature of the present development is that it includes the type 2a and 2b automorphisms, but no mention of these automorphisms (nor of any special cases of them) appears in Levstein's paper.

4. Properties of the automorphisms

4.1. Identical automorphisms and the identity automorphism

Suppose that the d_Γ-dimensional irreducible representation Γ of the simple Lie algebra $\tilde{\mathscr{L}}^0$ is such that the contragredient representation $-\tilde{\Gamma}$ is equivalent to the Γ, so that there exists a non-singular $d_\Gamma \times d_\Gamma$ matrix $\mathbf{C}$ such that

$$-\tilde{\Gamma}(a^0) = C^{-1}\Gamma(a^0)C \tag{105}$$

for all $a^0 \in \tilde{\mathscr{L}}^0$. Then it is obvious that the type 1b automorphism corresponding to the triple $\{C, u, \xi\}$ is identical to the type 1a automorphism corresponding to the triple $\{\mathbf{1}, u, \xi\}$, and also that the type 2b automorphism corresponding to the triple $\{C, u, \xi\}$ is identical to the type 2a automorphism corresponding to the triple $\{1, u, \xi\}$. Thus if $-\tilde{\Gamma}$ is equivalent to Γ there is essentially no distinction between automorphism of types 1a and 1b, nor between those of types 2a and 2b. Consequently, when this situation arises, attention will be concentrated solely on the type 1a and 2a automorphisms.

The type 1a automorphism ϕ of (67) is the identity automorphism if and only if

$$\xi = 0 \tag{106}$$

$$u = 1 \tag{107}$$

and

$$U(t) = \eta t^k \mathbf{1} \tag{108}$$

where η is any non-zero complex number and k is any integer. (It is easily checked that if these conditions are are satisfied then ϕ is the identity mapping. Conversely, if ϕ is the identity mapping, then it is necessary that

$$U(t)a(ut)U(t)^{-1} = a(t) \tag{109}$$

for all $a(t)$. In particular, with $a(t) = \Gamma(a^0)$, where $a^0 \in \tilde{\mathscr{L}}^0$, Schur's lemma implies that $U(t) = \eta(t)\mathbf{1}$ for some function $\eta(t)$. Then $U(t)^{-1} = (\eta(t))^{-1}\mathbf{1}$, and clearly both $\eta(t)$ and $(\eta(t))^{-1}$ can only be Laurent polynomials in t if $\eta(t) = \eta t^k$ for some complex number η and some integer k. Equation (109) then requires that $u=1$, while it is obviously necessary that $\xi=0$.)

Turning to the other types of automorphism, clearly a type 1b automorphism can be the identity mapping only if $-\tilde{\mathbf{\Gamma}}$ is equivalent to $\mathbf{\Gamma}$, which, as noted above, is the situation in which the type 1b automorphisms do not merit any separate study. Moreover, it is also clear from (71) and (72) that no type 2a or 2b automorphism can be the identity mapping.

The definitions given in (67)-(73) also imply that if the triples $\{\boldsymbol{U}(t), u, \xi\}$ and $\{\boldsymbol{U}'(t), u', \xi'\}$ specify two automorphisms of the same type, then these automorphisms are identical if and only if

$$u' = u \qquad \xi' = \xi \tag{110}$$

and there exists a non-zero complex number η and an integer k such that

$$\boldsymbol{U}'(t) = \eta t^k \boldsymbol{U}(t). \tag{111}$$

4.2. Products of automorphisms

(i) If ϕ_1 and ϕ_2 are two type 1a automorphisms corresponding to the triples $\{\boldsymbol{U}_1(t), u_1, \xi_1\}$ and $\{\boldsymbol{U}_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 1a automorphism corresponding to the triple $\{\boldsymbol{U}(t), u, \xi\}$, where

$$\boldsymbol{U}(t) = \boldsymbol{U}_1(t)\boldsymbol{U}_2(u_1 t) \tag{112}$$

$$u = u_1 u_2 \tag{113}$$

and

$$\xi = \xi_1 + \xi_2 + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(\boldsymbol{U}_1(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}_1(t)}{\mathrm{d}t}\boldsymbol{\Phi}(\boldsymbol{U}_2(u_1 t))\right)\right\} \tag{114}$$

and where $\boldsymbol{\Phi}(\boldsymbol{U}(t))$ is defined in (74).

(ii) If ϕ_1 and ϕ_2 are type 1a and 1b automorphisms corresponding to the triples $\{\boldsymbol{U}_1(t), u_1, \xi_1\}$ and $\{\boldsymbol{U}_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 1b automorphism corresponding to the triple $\{\boldsymbol{U}(t), u, \xi\}$, where $\boldsymbol{U}(t)$, u and ξ are again given by (112), (113) and (114) respectively.

(iii) If ϕ_1 and ϕ_2 are type 1a and 2a automorphisms corresponding to the triples $\{\boldsymbol{U}_1(t), u_1, \xi_1\}$ and $\{\boldsymbol{U}_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 2a automorphism corresponding to the triple $\{\boldsymbol{U}(t), u, \xi\}$, where

$$\boldsymbol{U}(t) = \boldsymbol{U}_1(t)\boldsymbol{U}_2(u_1 t) \tag{115}$$

$$u = u_1^{-1} u_2 \tag{116}$$

and

$$\xi = -\xi_1 + \xi_2 - \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(\boldsymbol{U}_1(t)^{-1}\frac{\mathrm{d}\boldsymbol{U}_1(t)}{\mathrm{d}t}\boldsymbol{\Phi}(\boldsymbol{U}_2(u_1 t))\right)\right\} \tag{117}$$

and where $\boldsymbol{\Phi}(\boldsymbol{U}(t))$ is defined in (74).

(iv) If ϕ_1 and ϕ_2 are type 1a and 2b automorphisms corresponding to the triples $\{\boldsymbol{U}_1(t), u_1, \xi_1\}$ and $\{\boldsymbol{U}_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 2b automorphism corresponding to the triple $\{\boldsymbol{U}(t), u, \xi\}$, where $\boldsymbol{U}(t)$, u and ξ are again given by (115), (116) and (117) respectively.

(v) If ϕ_1 and ϕ_2 are type 1b and 1a automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 1b automorphism corresponding to the triple $\{U(t), u, \xi\}$, where

$$U(t) = U_1(t)\tilde{U}_2(u_1 t)^{-1} \tag{118}$$

$$u = u_1 u_2 \tag{119}$$

and

$$\xi = \xi_1 + \xi_2 - \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U_1(t)^{-1}\frac{\mathrm{d}U_1(t)}{\mathrm{d}t}\tilde{\Phi}(U_2(u_1 t))\right)\right\} \tag{120}$$

and where $\Phi(U(t))$ is defined in (74).

(vi) If ϕ_1 and ϕ_2 are two type 1b automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 1a automorphism corresponding to the triple $\{U(t), u, \xi\}$, where $U(t)$, u and ξ are again given by (118), (119) and (120) respectively.

(vii) If ϕ_1 and ϕ_2 are type 1b and 2a automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 2b automorphism corresponding to the triple $\{U(t), u, \xi\}$, where

$$U(t) = U_1(t)\tilde{U}_2(u_1 t)^{-1} \tag{121}$$

$$u = u_1^{-1} u_2 \tag{122}$$

and

$$\xi = -\xi_1 + \xi_2 + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U_1(t)^{-1}\frac{\mathrm{d}U_1(t)}{\mathrm{d}t}\tilde{\Phi}(U_2(u_1 t))\right)\right\} \tag{123}$$

and where $\Phi(U(t))$ is defined in (74).

(vii) If ϕ_1 and ϕ_2 are type 1b and 2b automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 2a automorphism corresponding to the triple $\{U(t), u, \xi\}$, where $U(t)$, u and ξ are again given by (121), (122) and (123) respectively.

(ix) If ϕ_1 and ϕ_2 are type 2a and 1a automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 2a automorphism corresponding to the triple $\{U(t), u, \xi\}$, where

$$U(t) = U_1(t)U_2(u_1 t^{-1}) \tag{124}$$

$$u = u_1 u_2 \tag{125}$$

and

$$\xi = \xi_1 - \xi_2 + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U_1(t)^{-1}\frac{\mathrm{d}U_1(t)}{\mathrm{d}t}\Phi(U_2(u_1 t^{-1}))\right)\right\} \tag{126}$$

and where $\Phi(U(t))$ is defined in (74).

(x) If ϕ_1 and ϕ_2 are type 2a and 1b automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 2b automorphism corresponding to the triple $\{U(t), u, \xi\}$, where $U(t)$, u and ξ are again given by (124), (125) and (126) respectively.

(xi) If ϕ_1 and ϕ_2 are two type 2a automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 1a automorphism corresponding to the triple $\{U(t), u, \xi\}$, where

$$U(t) = U_1(t) U_2(u_1 t^{-1}) \tag{127}$$

$$u = u_1^{-1} u_2 \tag{128}$$

and

$$\xi = -\xi_1 - \xi_2 - \frac{1}{\gamma} \operatorname{Res}\left\{ \operatorname{tr}\left(U_1(t)^{-1} \frac{\mathrm{d} U_1(t)}{\mathrm{d}t} \Phi(U_2(u_1 t^{-1})) \right) \right\} \tag{129}$$

and where $\Phi(U(t))$ is defined in (74).

(xii) If ϕ_1 and ϕ_2 are type 2a and 2b automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 1b automorphism corresponding to the triple $\{U(t), u, \xi\}$, where $U(t)$, u and ξ are again given by (127), (128) and (129) respectively.

(xiii) If ϕ_1 and ϕ_2 are type 2b and 1a automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 2b automorphism corresponding to the triple $\{U(t), u, \xi\}$, where

$$U(t) = U_1(t) \tilde{U}_2(u_1 t^{-1})^{-1} \tag{130}$$

$$u = u_1 u_2 \tag{131}$$

and

$$\xi = \xi_1 - \xi_2 - \frac{1}{\gamma} \operatorname{Res}\left\{ \operatorname{tr}\left(U_1(t)^{-1} \frac{\mathrm{d} U_1(t)}{\mathrm{d}t} \tilde{\Phi}(U_2(u_1 t^{-1})) \right) \right\} \tag{132}$$

and where $\Phi(U(t))$ is defined in (74).

(xiv) If ϕ_1 and ϕ_2 are type 2b and 1b automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 2a automorphism corresponding to the triple $\{U(t), u, \xi\}$, where $U(t)$, u and ξ are again given by (130), (131) and (132) respectively.

(xv) If ϕ_1 and ϕ_2 are type 2b and 2a automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 1b automorphism corresponding to the triple $\{U(t), u, \xi\}$, where

$$U(t) = U_1(t) \tilde{U}_2(u_1 t^{-1})^{-1} \tag{133}$$

$$u = u_1^{-1} u_2 \tag{134}$$

and

$$\xi = -\xi_1 - \xi_2 + \frac{1}{\gamma} \operatorname{Res}\left\{ \operatorname{tr}\left(U_1(t)^{-1} \frac{\mathrm{d} U_1(t)}{\mathrm{d}t} \tilde{\Phi}(U_2(u_1 t^{-1})) \right) \right\} \tag{135}$$

and where $\Phi(U(t))$ is defined in (74).

(xvi) If ϕ_1 and ϕ_2 are two type 2b automorphisms corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 1a automorphism corresponding to the triple $\{U(t), u, \xi\}$, where $U(t)$, u and ξ are again given by (133), (134) and (135) respectively.

4.3. *Conditions for an automorphism to be involutive*

(i) It follows from (106)-(108) and (112)-(114) that a type 1a automorphism corresponding to the triple $\{U(t), u, \xi\}$ is involutive if and only if the following three conditions are all satisfied:

$$U(t)U(ut)=\eta t^k \mathbf{1} \tag{136}$$

for some complex number η and some integer k,

$$u^2=1 \tag{137}$$

and

$$\xi=-\frac{1}{2\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U(t)^{-1}\frac{\mathrm{d}U(t)}{\mathrm{d}t}\Phi(U(ut))\right)\right\}. \tag{138}$$

(ii) It follows from (106)-(108) and (118)-(120) that a type 1b automorphism corresponding to the triple $\{U(t), u, \xi\}$ is involutive if and only if the following three conditions are all satisfied:

$$U(t)\tilde{U}(ut)^{-1}=\eta t^k \mathbf{1} \tag{139}$$

for some complex number η and some integer k,

$$u^2=1 \tag{140}$$

and

$$\xi=\frac{1}{2\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U(t)^{-1}\frac{\mathrm{d}U(t)}{\mathrm{d}t}\tilde{\Phi}(U(ut))\right)\right\}. \tag{141}$$

(iii) It follows from (106)-(108) and (127)-(129) that a type 2a automorphism corresponding to the triple $\{U(t), u, \xi\}$ is involutive if and only if the following two conditions are satisfied:

$$U(t)U(ut^{-1})=\eta t^k \mathbf{1} \tag{142}$$

for some complex number η and some integer k, and

$$\xi=-\frac{1}{2\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U(t)^{-1}\frac{\mathrm{d}U(t)}{\mathrm{d}t}\Phi(U(ut^{-1}))\right)\right\}. \tag{143}$$

(iv) It follows from (106)-(108) and (133)-(135) that a type 2b automorphism corresponding to the triple $\{U(t), u, \xi\}$ is involutive if and only if the following two conditions are satisfied:

$$U(t)\tilde{U}(ut^{-1})^{-1}=\eta t^k \mathbf{1} \tag{144}$$

for some complex number η and some integer k, and

$$\xi=\frac{1}{2\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U(t)^{-1}\frac{\mathrm{d}U(t)}{\mathrm{d}t}\tilde{\Phi}(U(ut^{-1}))\right)\right\}. \tag{145}$$

4.4. *Inverses of automorphisms*

(i) It follows from equations (106)-(108) and (112)-(114) that the inverse of the type 1a automorphism corresponding to the triple $\{U(t), u, \xi\}$ is the type 1a automorphism corresponding to the triple $\{U'(t), u', \xi'\}$, where

$$U'(t)=U(u^{-1}t)^{-1} \tag{146}$$

$$u'=u^{-1} \tag{147}$$

and

$$\xi' = -\xi - \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U(t)^{-1}\frac{\mathrm{d}U(t)}{\mathrm{d}t}\Phi(U(t)^{-1})\right)\right\}. \tag{148}$$

(ii) It follows from equations (106)–(108) and (118)–(120) that the inverse of the type 1b automorphism corresponding to the triple $\{U(t), u, \xi\}$ is the type 1b automorphism corresponding to the triple $\{U'(t), u', \xi'\}$, where

$$U'(t) = \tilde{U}(u^{-1}t) \tag{149}$$

$$u' = u^{-1} \tag{150}$$

and

$$\xi' = -\xi + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U(t)^{-1}\frac{\mathrm{d}U(t)}{\mathrm{d}t}\tilde{\Phi}(\tilde{U}(t))\right)\right\}. \tag{151}$$

(iii) It follows from equations (106)–(108) and (127)–(129) that the inverse of the type 2a automorphism corresponding to the triple $\{U(t), u, \xi\}$ is the type 2a automorphism corresponding to the triple $\{U'(t), u', \xi'\}$, where

$$U'(t) = U(ut^{-1})^{-1} \tag{152}$$

$$u' = u \tag{153}$$

and

$$\xi' = -\xi - \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U(t)^{-1}\frac{\mathrm{d}U(t)}{\mathrm{d}t}\Phi(U(t)^{-1})\right)\right\}. \tag{154}$$

(iv) It follows from equations (106)–(108) and (133)–(135) that the inverse of the type 2b automorphism corresponding to the triple $\{U(t), u, \xi\}$ is the type 2b automorphism corresponding to the triple $\{U'(t), u', \xi'\}$, where

$$U'(t) = \tilde{U}(ut^{-1}) \tag{155}$$

$$u' = u \tag{156}$$

and

$$\xi' = -\xi + \frac{1}{\gamma}\operatorname{Res}\left\{\operatorname{tr}\left(U(t)^{-1}\frac{\mathrm{d}U(t)}{\mathrm{d}t}\tilde{\Phi}(\tilde{U}(t))\right)\right\}. \tag{157}$$

4.5. Conjugacy conditions for automorphisms

The necessary and sufficient conditions for the conjugacy of a pair of automorphisms ϕ_1 and ϕ_2 of $\tilde{\mathscr{L}}$ corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively via an automorphism ϕ of $\tilde{\mathscr{L}}$ corresponding to the triple $\{S(t), s, \xi\}$ will now be investigated. More precisely, these are the necessary and sufficient conditions for the automorphism equality $\phi_1 = \phi \circ \phi_2 \circ \phi^{-1}$ to hold. In each case there are three such conditions, but only two will be exhibited explicitly. These are the conditions relating the matrix $U_1(t)$ to the matrix $U_2(t)$ and the parameter u_1 to the parameter u_2. It is possible also to relate the parameter ξ_1 to the parameter ξ_2, but the resulting expressions will be omitted as they are very complicated and will not be needed in the subsequent analysis.

It will be obvious in every case that if ϕ_1 and ϕ_2 are conjugate then they must be of the same type.

4.5.1. Conditions for the conjugacy of two type 1a automorphisms

(i) If ϕ_1 and ϕ_2 are two type 1a automorphisms and ϕ is a type 1a automorphism, the conditions are

$$\eta t^k \boldsymbol{U}_1(t) = \boldsymbol{S}(t)\boldsymbol{U}_2(st)\boldsymbol{S}(u_2 t)^{-1} \tag{158}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = u_2. \tag{159}$$

(ii) If ϕ_1 and ϕ_2 are two type 1a automorphisms and ϕ is a type 1b automorphism, the conditions are

$$\eta t^k \boldsymbol{U}_1(t) = \boldsymbol{S}(t)\tilde{\boldsymbol{U}}_2(st)^{-1}\boldsymbol{S}(u_2 t)^{-1} \tag{160}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = u_2. \tag{161}$$

(iii) If ϕ_1 and ϕ_2 are two type 1a automorphisms and ϕ is a type 2a automorphism, the conditions are

$$\eta t^k \boldsymbol{U}_1(t) = \boldsymbol{S}(t)\boldsymbol{U}_2(st^{-1})\boldsymbol{S}(u_2^{-1} t)^{-1} \tag{162}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = u_2^{-1}. \tag{163}$$

(iv) If ϕ_1 and ϕ_2 are two type 1a automorphisms and ϕ is a type 2b automorphism, the conditions are

$$\eta t^k \boldsymbol{U}_1(t) = \boldsymbol{S}(t)\tilde{\boldsymbol{U}}_2(st^{-1})^{-1}\boldsymbol{S}(u_2^{-1} t)^{-1} \tag{164}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = u_2^{-1}. \tag{165}$$

4.5.2. Conditions for the conjugacy of two type 1b automorphisms

(i) If ϕ_1 and ϕ_2 are two type 1b automorphisms and ϕ is a type 1a automorphism, the conditions are

$$\eta t^k \boldsymbol{U}_1(t) = \boldsymbol{S}(t)\boldsymbol{U}_2(st)\tilde{\boldsymbol{S}}(u_2 t) \tag{166}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = u_2. \tag{167}$$

(ii) If ϕ_1 and ϕ_2 are two type 1b automorphisms and ϕ is a type 1b atuomorphism, the conditions are

$$\eta t^k \boldsymbol{U}_1(t) = \boldsymbol{S}(t)\tilde{\boldsymbol{U}}_2(st)^{-1}\tilde{\boldsymbol{S}}(u_2 t) \tag{168}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = u_2. \tag{169}$$

(iii) If ϕ_1 and ϕ_2 are two type 1b automorphisms and ϕ is a type 2a automorphism, the conditions are

$$\eta t^k \boldsymbol{U}_1(t) = \boldsymbol{S}(t)\boldsymbol{U}_2(st^{-1})\tilde{\boldsymbol{S}}(u_2^{-1} t) \tag{170}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = u_2^{-1}. \tag{171}$$

(iv) If ϕ_1 and ϕ_2 are two type 1b automorphisms and ϕ is a type 2b automorphism, the conditions are

$$\eta t^k U_1(t) = S(t)\tilde{U}_2(st^{-1})^{-1}\tilde{S}(u_2^{-1}t) \tag{172}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = u_2^{-1}. \tag{173}$$

4.5.3. Conditions for the conjugacy of two type 2a automorphisms

(i) If ϕ_1 and ϕ_2 are two type 2a automorphisms and ϕ is a type 1a automorphism, the conditions are

$$\eta t^k U_1(t) = S(t) U_2(st) S(s^{-2}u_2 t^{-1})^{-1} \tag{174}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = s^{-2}u_2. \tag{175}$$

(ii) If ϕ_1 and ϕ_2 are two type 2a automorphisms and ϕ is a type 1b automorphism, the conditions are

$$\eta t^k U_1(t) = S(t)\tilde{U}_2(st)^{-1} S(s^{-2}u_2 t^{-1})^{-1} \tag{176}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = s^{-2}u_2. \tag{177}$$

(iii) If ϕ_1 and ϕ_2 are two type 2a automorphisms and ϕ is a type 2a automorphism, the conditions are

$$\eta t^k U_1(t) = S(t) U_2(st^{-1}) S(s^2 u_2^{-1} t^{-1})^{-1} \tag{178}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = s^2 u_2^{-1}. \tag{179}$$

(iv) If ϕ_1 and ϕ_2 are two type 2a automorphisms and ϕ is a type 2b automorphism, the conditions are

$$\eta t^k U_1(t) = S(t)\tilde{U}_2(st^{-1})^{-1} S(s^2 u_2^{-1} t^{-1})^{-1} \tag{180}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = s^2 u_2^{-1}. \tag{181}$$

4.5.4. Conditions for the conjugacy of two type 2b automorphisms

(i) If ϕ_1 and ϕ_2 are two type 2b automorphisms and ϕ is a type 1a automorphism, the conditions are

$$\eta t^k U_1(t) = S(t) U_2(st)\tilde{S}(s^{-2}u_2 t^{-1}) \tag{182}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = s^{-2}u_2. \tag{183}$$

(ii) If ϕ_1 and ϕ_2 are two type 2b automorphisms and ϕ is a type 1b automorphism, the conditions are

$$\eta t^k U_1(t) = S(t)\tilde{U}_2(st)^{-1}\tilde{S}(s^{-2}u_2 t^{-1}) \tag{184}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = s^{-2}u_2. \tag{185}$$

(iii) If ϕ_1 and ϕ_2 are two type 2b automorphisms and ϕ is a type 2a automorphism, the conditions are

$$\eta t^k U_1(t) = S(t) U_2(st^{-1}) \tilde{S}(s^2 u_2^{-1} t^{-1}) \tag{186}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = s^2 u_2^{-1}. \tag{187}$$

(iv) If ϕ_1 and ϕ_2 are two type 2b automorphisms and ϕ is a type 2b automorphism, the conditions are

$$\eta t^k U_1(t) = S(t) \tilde{U}_2(st^{-1})^{-1} \tilde{S}(s^2 u_2^{-1} t^{-1}) \tag{188}$$

where η is any non-zero complex number and k is any integer, and

$$u_1 = s^2 u_2^{-1}. \tag{189}$$

4.6. General remarks on the conjugacy classes of involutive automorphisms

One of the main motivations of the preceding analysis is to set up the means for determining the conjugacy classes of involutive automorphisms of $\tilde{\mathscr{L}}$, the aim being to specify one representative in each such conjugacy class.

Consider first the case of involutive automorphisms of type 1a. For an involutive automorphism of type 1a the condition (137) implies that the only allowed values of the parameter u are such that

$$u = 1 \text{ or } -1. \tag{190}$$

However, relations (159), (161), (163) and (165) show that the value of u is left invariant under all conjugacy transformations, so every class of type 1a involutive automorphisms with $u = 1$ is disjoint from every class of type 1a involutive automorphisms with $u = -1$. Thus in enumerating the conjugacy classes of type 1a involutive automorphisms it is necessary to consider the two cases $u = 1$ and $u = -1$ separately.

Exactly the same comments apply to the case of involutive automorphisms of type 1b.

Now consider the case of involutive automorphisms of type 2a. For these there are no constraints like (190), but (175), (177), (179) and (181) imply that every value of u is attainable in every conjugacy class. Consequently one may always *choose* the representative of each conjugacy class of type 2a involutive automorphisms to correspond to

$$u = 1. \tag{191}$$

For the case of involutive automorphisms of type 2b the situation is exactly the same as for involutive automorphisms of type 2a, so one may again always *choose* the representative of each conjugacy class of type 2b involutive automorphisms to correspond to

$$u = 1. \tag{192}$$

Of course, as implied by the observations at the beginning of this section, if the d_Γ-dimensional irreducible representation Γ of the simple Lie algebra $\tilde{\mathscr{L}}^0$ is such that the contragredient representation $-\tilde{\Gamma}$ is equivalent to Γ, so that there exists a non-singular $d_\Gamma \times d_\Gamma$ matrix C such that (105) holds, then the classes of type 1b involutive automorphisms of $\tilde{\mathscr{L}}$ coincide with the classes of type 1a involutive automorphisms of $\tilde{\mathscr{L}}$, and the classes of type 2b involutive automorphisms of $\tilde{\mathscr{L}}$ coincide with the classes of type 2a involutive automorphisms of $\tilde{\mathscr{L}}$. That is, in such a situation it is only necessary to investigate explicitly the conjugacy classes of type 1a involutive automorphisms and of type 2a involutive automorphisms.

References

[1] Gantmacher F R 1939 *Rec. Math.* (*Mat. Sbornik*) NS **5**(47) 101–44
[2] Gantmacher F R 1939 *Rec. Math.* (*Mat. Sbornik*) NS **5**(47) 217–50
[3] Kac V G 1985 *Infinite Dimensional Lie Algebras* (Cambridge: Cambridge University Press)
[4] Cornwell J F 1989 *Group Theory in Physics* vol III (London: Academic Press)
[5] Gorman N, O'Raifeartaigh L and McGlinn W 1989 *J. Math. Phys.* **30** 1921–32
[6] Kac V G and Wakimoto M 19888 *Adv. Math.* **70** 156–236
[7] Frenkel I B and Kac V G 1980 *Invent. Math.* **62** 23–66
[8] Levstein F 1988 *J. Alg.* **114** 489–518